

CHROMATIC QUASISYMMETRIC FUNCTIONS AND HESSENBERG VARIETIES

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ABSTRACT. We discuss three distinct topics of independent interest; one in enumerative combinatorics, one in symmetric function theory, and one in algebraic geometry. The topic in enumerative combinatorics concerns a q -analog of a generalization of the Eulerian polynomials, the one in symmetric function theory deals with a refinement of the chromatic symmetric functions of Stanley, and the one in algebraic geometry deals with Tymoczko's representation of the symmetric group on the cohomology of the regular semisimple Hessenberg variety of type A. Our purpose is to explore some remarkable connections between these topics.

CONTENTS

1. Introduction	1
2. q -Eulerian numbers and toric varieties	6
3. Rawlings major index and generalized Eulerian numbers	11
4. Chromatic quasisymmetric functions	15
5. Hessenberg varieties	21
Acknowledgments	24
References	24

1. INTRODUCTION

Let $H(z) := \sum_{n \geq 0} h_n z^n$, where h_n is the complete homogeneous symmetric function of degree n . The formal power series

$$(1.1) \quad \frac{(1-t)H(z)}{H(tz) - tH(z)},$$

has occurred in various contexts in the literature, including

- the study of a class of q -Eulerian polynomials

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- the enumeration of Smirnov words by descent number
- the study of the toric variety associated with the Coxeter complex of the symmetric group.

We briefly describe these contexts now and give more detail in Section 2 with definitions in Section 1.1. In [ShWa1, ShWa2] Shreshian and Wachs initiated a study of the joint distribution of the Mahonian permutation statistic, major index, and the Eulerian permutation statistic, excedance number. They showed that the q -Eulerian polynomial defined by

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} t^{\text{exc}(\sigma)}$$

can be obtained by taking a certain specialization of the coefficient of z^n in (1.1) known as the stable principal specialization. This enabled them to derive the q -analog of a formula of Euler formula that is given in (2.10).

The work in [ShWa1] led Stanley to derive a refinement of a formula of Carlitz, Scoville and Vaughan [CaScVa] for the enumerator of words over the alphabet \mathbb{P} with no adjacent repeats (see [ShWa2]). These are known as Smirnov words. In Stanley's refinement the Smirnov words are enumerated according to their number of descents and the enumerator is equal to the symmetric function obtained by applying the standard involution ω on the ring of symmetric functions to (1.1).

In another direction, Stanley [St2, Proposition 7.7] used a formula of Procesi [Pr] to show that (1.1) is the generating function for the Frobenius characteristic of the representation of the symmetric group \mathfrak{S}_n on the cohomology of the toric variety associated with the Coxeter complex of type A_{n-1} .

The q -Eulerian polynomial, the Smirnov word enumerator, and the \mathfrak{S}_n -representation on the cohomology of the toric variety all have very nice generalizations. In this paper we discuss these generalizations and explore the relationships among these generalizations and the ramifications of the relationship. The proofs of the results surveyed here either have appeared in [ShWa2] or will appear in [ShWa4].

The generalization of the q -Eulerian numbers, as defined in [ShWa4], is obtained by considering the joint distribution of a Mahonian statistic of Rawlings [Ra] that interpolates between inversion number and major index, and a generalized Eulerian statistic of De Mari and Shayman [DeSh]. We discuss these generalized q -Eulerian numbers and a more general version of them associated with posets in Section 3.

The generalization of the Smirnov word enumerator, as defined in [ShWa4], is a refinement of Stanley's well known chromatic symmetric

function [St6], which we call a chromatic quasisymmetric function. In Section 3 we report on results on the chromatic quasisymmetric functions, which are proved in [ShWa4]. In particular we present a refinement of Gasharov's [Ga] expansion in the Schur basis and a refinement of Chow's [Ch] expansion in the basis of fundamental quasisymmetric functions. We also present a conjectured refinement of Stanley's [St5] expansion in the power-sum basis and a refinement of Stanley's [St5] conjecture on e -positivity.

The generalization of the toric variety associated with the type A Coxeter complex is the regular semisimple Hessenberg variety of type A first studied by De Mari and Shayman [DeSh] and further studied by De Mari, Procesi and Shayman [DePrSh]. Tymoczko [Ty2] defined a representation of the symmetric group on the cohomology of this Hessenberg variety, which generalizes the representation studied by Procesi and Stanley. This generalization is described in more detail in Section 5.

In Section 4 we present the following extension of the relation between q -Eulerian polynomials and Smirnov words. As noted above, we can associate to a finite poset P a generalized q -Eulerian polynomial, which will be denoted by $A_P(q, t)$. Let $X_G(\mathbf{x}, t)$ denote the chromatic quasisymmetric function of a graph G .

Theorem 1.1. [ShWa4] *Let G be the incomparability graph of a poset P on $[n]$. Then*

$$A_P(q, t) = (q; q)_n \mathbf{ps}(\omega X_G(\mathbf{x}, t)),$$

where \mathbf{ps} denotes stable principal specialization.

In Section 5 we discuss the following conjectured generalization of relationship between Smirnov words and the representation of the symmetric group on the cohomology of the toric variety.

Conjecture 1.2. [ShWa4] *Let G be the incomparability graph of a natural unit interval order P . Then*

$$\omega X_G(\mathbf{x}, t) = \sum_{j=0}^{|E(G)|} \text{ch} H^{2j}(\mathcal{H}(P)) t^j,$$

where ch denotes the Frobenius characteristic and $H^{2j}(\mathcal{H}(P))$ denotes Tymoczko's representation of \mathfrak{S}_n on the degree $2j$ cohomology of the Hessenberg variety $\mathcal{H}(P)$ associated with P .

This conjecture has the potential of providing solutions to several open problems such as,

- (1) *Sharesian and Wachs' conjecture that the generalized q -Eulerian polynomials are unimodal* [ShWa4]. This would follow from Theorem 1.1 and the hard Lefschetz theorem applied to Tymoczko's representation on the cohomology of the Hessenberg variety.
- (2) *Tymoczko's problem of finding a decomposition of her representation into irreducibles* [Ty2]. Such a decomposition would be provided by the expansion of the chromatic quasisymmetric functions in the Schur function basis given in Theorem 4.11.
- (3) *Stanley's well-known conjecture that the chromatic symmetric functions (for unit interval orders) are e -positive* [St5]. This would be equivalent to Conjecture 5.4 that Tymoczko's representation is a permutation representation in which each point stabilizer is a Young subgroup.

1.1. Preliminaries. We assume that the reader is familiar with some basic notions from combinatorics and symmetric function theory. All terms used but not defined in this paper are defined in [St3] or [St4].

Let $\sigma \in \mathfrak{S}_n$, where \mathfrak{S}_n denotes the symmetric group on $[n] := \{1, 2, \dots, n\}$. The excedance number of σ is given by

$$\text{exc}(\sigma) := |\{i \in [n-1] : \sigma(i) > i\}|.$$

The descent set of σ is given by

$$\text{DES}(\sigma) := \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$$

and the descent number and major index are

$$\text{des}(\sigma) := |\text{DES}(\sigma)| \quad \text{and} \quad \text{maj}(\sigma) := \sum_{i \in \text{DES}(\sigma)} i.$$

It is well-known that the permutation statistics exc and des are equidistributed on \mathfrak{S}_n . The common generating functions for these statistics are called Eulerian polynomials. That is, the Eulerian polynomials are defined as

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)}.$$

Any permutation statistic with generating function $A_n(t)$ is called an Eulerian statistic. The Eulerian numbers are the coefficients of the Eulerian polynomials; they are defined as

$$a_{n,j} := \{\sigma \in \mathfrak{S}_n : \text{des}(\sigma) = j\} = \{\sigma \in \mathfrak{S}_n : \text{exc}(\sigma) = j\},$$

for $0 \leq j \leq n-1$.

It is also well-known that the major index is equidistributed with the inversion index defined as

$$\text{inv}(\sigma) = |\{(i, j) \in [n] \times [n] : i < j \text{ and } \sigma(i) > \sigma(j)\}|$$

and that

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} = [n]_q!,$$

where

$$[n]_q := 1 + q + \cdots + q^{n-1} \text{ and } [n]_q! := [n]_q [n-1]_q \cdots [1]_q.$$

Any permutation statistic equidistributed with maj and inv is called a Mahonian permutation statistic.

Let R be a ring with a partial order relation \leq . Given a sequence (a_0, a_1, \dots, a_n) of elements of R we say that the sequence is *palindromic* with center of symmetry $\frac{n}{2}$ if $a_j = a_{n-j}$ for $0 \leq j \leq n$. The sequence is said to be *unimodal* if

$$a_0 \leq a_1 \leq \cdots \leq a_c \geq a_{c+1} \geq a_{c+2} \geq \cdots \geq a_n$$

for some c . The sequence is said to be *positive* if $a_i \geq 0$ for each i . We say that the polynomial $A(t) := a_0 + a_1 t + \cdots + a_n t^n \in R[t] - \{0\}$ is positive, palindromic and unimodal with center of symmetry $\frac{n}{2}$ if (a_0, a_1, \dots, a_n) is positive, palindromic and unimodal, respectively. We will use the following basic tool.

Proposition 1.3 (see [St2, Proposition 1]). *Let $A(t)$ and $B(t)$ be positive, palindromic and unimodal polynomials in $R[t]$ with respective centers of symmetry c_A and c_B . Then*

- (1) $A(t)B(t)$ is positive, palindromic and unimodal with center of symmetry $c_A + c_B$.
- (2) If $c_A = c_B$ then $A(t) + B(t)$ is positive, palindromic and unimodal with center of symmetry c_A .

From Proposition 1.3 (1) we see that $[n]_t!$ is positive, palindromic and unimodal with center of symmetry $\frac{n(n+1)}{4}$.

For any ring R , let Λ_R be the ring of symmetric functions in variables x_1, x_2, \dots with coefficients in R . Let b be any basis for $\Lambda_{\mathbb{Z}}$. If f is a symmetric function whose expansion in b has nonnegative coefficients, we say that f is b -positive. This induces a partial order relation on the ring $\Lambda_{\mathbb{Z}}$ given by $f \leq_b g$ if $g - f$ is b -positive. We use this partial order to define b -unimodality of sequences of symmetric functions in $\Lambda_{\mathbb{Z}}$ and polynomials in $\Lambda_{\mathbb{Z}}[t]$. Similarly, if we say that a sequence with entries in $\mathbb{Z}[q]$ or a polynomial in $\mathbb{Z}[q][t]$ is positive or unimodal, we are using

the partial order relation on $\mathbb{Z}[q]$ defined by $f(q) \leq g(q)$ if $g(q) - f(q)$ has nonnegative coefficients.

The bases for $\Lambda_{\mathbb{Z}}$ that are relevant here are the Schur basis $(s_{\lambda})_{\lambda \in \text{Par}}$, the h-basis $(h_{\lambda})_{\lambda \in \text{Par}}$, the e-basis $(e_{\lambda})_{\lambda \in \text{Par}}$, and the p-basis $(z^{-1}p_{\lambda})_{\lambda \in \text{Par}}$, where Par is the set of all integer partitions,

$$z_{\lambda} := \prod_i m_i(\lambda)! i^{m_i(\lambda)}$$

and $m_i(\lambda)$ is the number of parts of λ equal to i .

Recall that the Frobenius characteristic ch is a ring isomorphism from the ring of virtual representations of the symmetric groups to $\Lambda_{\mathbb{Z}}$. It takes the irreducible Specht module S^{λ} to the Schur function s_{λ} . As is customary, we use ω to denote the involution on $\Lambda_{\mathbb{Z}}$ that takes h_i to e_i . Recall $\omega(s_{\lambda}) = s_{\lambda'}$ for all $\lambda \in \text{Par}$, where λ' denotes the conjugate shape.

For any ring R , let \mathcal{Q}_R be the ring of quasisymmetric functions in variables x_1, x_2, \dots with coefficients in R . For $n \in \mathbb{P}$ and $S \subseteq [n-1]$, let $D(S)$ be the set of all functions $f : [n] \rightarrow \mathbb{P}$ such that

- $f(i) \geq f(i+1)$ for all $i \in [n-1]$, and
- $f(i) > f(i+1)$ for all $i \in S$.

Recall Gessel's *fundamental quasisymmetric function*

$$F_{n,S} := \sum_{f \in D(S)} x_f,$$

where $x_f := x_{f(1)}x_{f(2)}\dots$. It is straightforward to confirm that $F_{n,S} \in \mathcal{Q}_{\mathbb{Z}}$. In fact (see [St4, Proposition 7.19.1]), $\{F_{n,S} : S \subseteq [n-1]\}$ is a basis for the free \mathbb{Z} -module of homogeneous degree n quasisymmetric functions with coefficients in \mathbb{Z} .

See [St4] for further information on symmetric functions, quasisymmetric functions and representations.

2. q -EULERIAN NUMBERS AND TORIC VARIETIES

It is well known that the Eulerian numbers $(a_{n,0}, a_{n,1}, \dots, a_{n,n-1})$ form the h -vector of the Coxeter complex Δ_n of the symmetric group \mathfrak{S}_n . Danilov and Jurciewicz (see [St2, eq. (26)]) showed that the j th entry of the h -vector of any convex simplicial polytope P is equal to $\dim H^{2j}(\mathcal{V}(P))$ where $\mathcal{V}(P)$ is the toric variety associated with P and $H^i(\mathcal{V}(P))$ is the degree i singular cohomology of $\mathcal{V}(P)$ over \mathbb{C} . (In odd degrees cohomology vanishes.) It follows that

$$(2.1) \quad a_{n,j} = \dim H^{2j}(\mathcal{V}(\Delta_n)),$$

for all $j = 0, \dots, n-1$.

It is also well known that the Eulerian numbers are palindromic and unimodal. Although there are elementary ways to prove this result, palindromicity and unimodality can be viewed as consequences of the hard Lefschetz theorem applied to the toric variety $\mathcal{V}(\Delta_n)$.

The classical hard Lefschetz theorem states that if \mathcal{V} is a projective smooth variety of complex dimension n then there is a linear operator ϕ on $H^*(\mathcal{V})$ sending each $H^i(\mathcal{V})$ to $H^{i+2}(\mathcal{V})$ such that for $0 \leq i \leq 2n-2$, the map $\phi^{n-i} : H^i(\mathcal{V}) \rightarrow H^{2n-i}(\mathcal{V})$ is a bijection. It follows from this that the sequence of even degree Betti numbers and the sequence of odd degree Betti numbers are palindromic and unimodal. Since $\mathcal{V}(\Delta_n)$ is a smooth variety, it follows that the Eulerian numbers form a palindromic and unimodal sequence. See [St1, St2] for further information on the use of the hard Lefschetz theorem in combinatorics.

2.1. Action of the symmetric group. The symmetric group \mathfrak{S}_n acts naturally on the Coxeter complex Δ_n and this induces a representation of \mathfrak{S}_n on cohomology of the toric variety $\mathcal{V}(\Delta_n)$. Procesi [Pr] derived a recursive formula for this representation, which Stanley [St2, see Proposition 12] used to obtain the following formula for the Frobenius characteristic of the representation,

$$(2.2) \quad 1 + \sum_{n \geq 1} \sum_{j=0}^{n-1} \text{ch} H^{2j}(\mathcal{V}(\Delta_n)) t^j z^n = \frac{(1-t)H(z)}{H(tz) - tH(z)}.$$

(See [Ste1, Ste2, DoLu, Le] for related work on this representation.) This is a symmetric function analog of the classical formula of Euler,

$$(2.3) \quad 1 + \sum_{n \geq 1} \sum_{j=0}^{n-1} a_{n,j} t^j \frac{z^n}{n!} = \frac{(1-t)e^z}{e^{tz} - te^z}.$$

An immediate consequence of (2.2) is that $\text{ch} H^{2j}(\mathcal{V}(\Delta_n))$ is h -positive, which is equivalent to saying that the linear representation of \mathfrak{S}_n on $H^{2j}(\mathcal{V}(\Delta_n))$ is obtained from an action on some set in which each point stabilizer is a Young subgroup. It is also easy to obtain the following closed form formula from (2.2).

Corollary 2.1 (see [ShWa2, Corollary 4.2]). *For all $n \geq 1$,*

$$(2.4) \quad \sum_{j=0}^{n-1} \text{ch} H^{2j}(\mathcal{V}(\Delta_n)) t^j = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{k_0 \geq 0 \\ k_1, \dots, k_m \geq 2 \\ \sum k_i = n}} h_{k_0} \prod_{i=1}^m h_{k_i} t[k_i - 1]_t.$$

Formula (2.2) has a number of additional interesting, but not so obvious, consequences. We discuss some of them below.

Stembridge [Ste1] uses (2.2) to characterize the multiplicity of each irreducible \mathfrak{S}_n -module in $H^{2j}(\mathcal{V}(\Delta_n))$ in terms of marked tableaux. A *marked tableau* of shape λ is a semistandard tableau of shape λ with entries in $\{0, 1, \dots, k\}$ for some $k \in \mathbb{N}$, such that each $a \in [k]$ occurs at least twice in the tableau, together with a mark on one occurrence of each $a \in [k]$; the marked occurrence cannot be the leftmost occurrence. The index of a marked a is the number of occurrences of a to the left of the marked a . The index of a marked tableau is the sum of the indices of the marked entries of the tableau.

Theorem 2.2 (Stembridge [Ste1]). *Let $\lambda \vdash n$. The multiplicity of the irreducible Specht module S^λ in $H^{2j}(\mathcal{V}(\Delta_n))$ is the number of marked tableaux of shape λ and index j .*

Recently, Shareshian and Wachs [ShWa4] obtained a different formula for the multiplicity of S^λ in terms of a different type of tableau (see Corollary 4.12).

The character of $H^{2j}(\mathcal{V}(\Delta_n))$ can be obtained from the following expansion in the p -basis.

Theorem 2.3 (Stembridge [Ste1], Dolgachev and Lunts [DoLu]). *For all $n \geq 1$,*

$$\sum_{j=0}^{n-1} \text{ch} H^{2j}(\mathcal{V}(\Delta_n)) t^j = \sum_{\lambda \vdash n} A_{l(\lambda)}(t) \prod_i [\lambda_i]_t z_\lambda^{-1} p_\lambda$$

2.2. Expansion in the basis of fundamental quasisymmetric functions. In their study of q -Eulerian numbers [ShWa1, ShWa2], Shareshian and Wachs obtain a formula for the expansion of the right hand side of (2.2) in the basis of fundamental quasisymmetric functions. In order to describe this decomposition the notion of DEX set of a permutation is needed. For $n \geq 1$, we set

$$[\overline{n}] := \{\overline{1}, \dots, \overline{n}\}$$

and totally order the alphabet $[n] \cup [\overline{n}]$ by

$$(2.5) \quad \overline{1} < \dots < \overline{n} < 1 < \dots < n.$$

For a permutation $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$, we define $\overline{\sigma}$ to be the word over alphabet $[n] \cup [\overline{n}]$ obtained from σ by replacing σ_i with $\overline{\sigma_i}$ whenever $i \in \text{EXC}(\sigma)$. For example, if $\sigma = 531462$ then $\overline{\sigma} = \overline{5}314\overline{6}2$. We define a descent in a word $w = w_1 \dots w_n$ over any totally ordered alphabet to

be any $i \in [n-1]$ such that $w_i > w_{i+1}$ and let $\text{DES}(w)$ be the set of all descents of w . Now, for $\sigma \in \mathfrak{S}_n$, we define

$$\text{DEX}(\sigma) := \text{DES}(\overline{\sigma}).$$

For example, $\text{DEX}(531462) = \text{DES}(\overline{531462}) = \{1, 4\}$.

Theorem 2.4 (Shareshian and Wachs [ShWa2, Theorem 1.2]).

$$(2.6) \quad 1 + \sum_{n \geq 1} \sum_{\sigma \in \mathfrak{S}_n} F_{\text{DEX}(\sigma), n} t^{\text{exc}(\sigma)} z^n = \frac{(1-t)H(z)}{H(tz) - tH(z)}.$$

Corollary 2.5 ([ShWa2, Theorem 7.4]). *For $0 \leq j \leq n-1$,*

$$\text{ch} H^{2j}(\mathcal{V}(\Delta_n)) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j}} F_{\text{DEX}(\sigma), n}$$

Expansion in the basis of fundamental quasisymmetric functions is useful because it can yield interesting results about permutation statistics via specialization. For a quasisymmetric function $Q(x_1, x_2, \dots) \in \mathcal{Q}_R$, the stable principal specialization $\mathbf{ps}(Q) \in R[[q]]$ is, by definition, obtained from Q by substituting q^{i-1} for x_i for each $i \in \mathbb{P}$. Now (see for example [St6, Lemma 7.19.10]), for any $n \in \mathbb{P}$ and $S \subseteq [n-1]$, we have

$$(2.7) \quad \mathbf{ps}(F_{n,S}) = (q; q)_n^{-1} q^{\sum_{i \in S} i},$$

where

$$(p; q)_n = \prod_{j=1}^n (1 - pq^{j-1}).$$

From Corollary 2.5 and the fact that $\sum_{i \in \text{DEX}(\sigma)} i = \text{maj}(\sigma) - \text{exc}(\sigma)$ (see [ShWa2, Lemma 2.2]) we obtain the following q -analog of (2.1).

Corollary 2.6. *For $0 \leq j \leq n-1$,*

$$(q; q)_n \mathbf{ps}(\text{ch} H^{2j}(\mathcal{V}(\Delta_n))) = a_{n,j}(q)$$

where

$$(2.8) \quad a_{n,j}(q) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{exc}(\sigma) = j}} q^{\text{maj}(\sigma) - \text{exc}(\sigma)}.$$

The q -Eulerian numbers $a_{n,j}(q)$ defined in Corollary 2.6 were initially studied in [ShWa1, ShWa2] and have been further studied in [ShWa3, FoHa1, FoHa2, SaShWa, HeWa]. Define the q -Eulerian polynomials,

$$(2.9) \quad A_n(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} t^{\text{exc}(\sigma)}.$$

In [ShWa1, ShWa2], the following q -analog of Euler's formula (2.3) was obtained via specialization of (2.6),

$$(2.10) \quad 1 + \sum_{n \geq 1} A_n(q, t) \frac{z^n}{[n]_q!} = \frac{(1-t) \exp_q(z)}{\exp_q(tz) - t \exp_q(z)}.$$

It is surprising that when one evaluates $a_{n,j}(q)$ at any n th root of unity, one always gets a positive integer.

Theorem 2.7 (Sagan, Shareshian, and Wachs [SaShWa, Corollary 6.2]). *Let $dm = n$ and let ξ_d be any primitive d th root of unity. Then*

$$A_n(\xi_d, t) = A_m(t) [d]_t^m.$$

Consequently $A_n(\xi_d, t)$ is positive, palindromic, unimodal polynomial in $\mathbb{Z}[t]$.

Theorem 2.3, Corollary 2.6 and the following result which is implicit in [De] and stated explicitly in [SaShWa], can be used to give a proof of Theorem 2.7.

Lemma 2.8. *Suppose $f(q) \in \mathbb{Z}[q]$ and there exists a homogeneous symmetric function F of degree n with coefficients in \mathbb{Z} such that*

$$f(q) = (q; q)_n \mathbf{ps}(F).$$

If $dm = n$ then $f(\xi_d)$ is the coefficient of $z_{d^m}^{-1} p_{d^m}$ in the expansion of F in the power-sum basis.

2.3. Unimodality. Let us now consider what the hard Lefschetz theorem tells us about the sequence $(\mathrm{ch} H^{2j}(\mathcal{V}(\Delta_n)))_{0 \leq j \leq n-1}$. Since the action of the symmetric group \mathfrak{S}_n commutes with the hard Lefschetz map ϕ (see [St2, p. 528]), we can conclude that for $0 \leq i \leq n$ the map $\phi : H^i(\mathcal{V}(\Delta_n)) \rightarrow H^{i+2}(\mathcal{V}(\Delta_n))$ is a \mathfrak{S}_n -module monomorphism. Hence, by Schur's lemma, for each $\lambda \vdash n$, the multiplicity of the Specht module S^λ in $H^i(\mathcal{V}(\Delta_n))$ is less than or equal to the multiplicity in $H^{i+2}(\mathcal{V}(\Delta_n))$. Equivalently, the coefficient of the Schur function s_λ in the expansion of $\mathrm{ch} H^i(\mathcal{V}(\Delta_n))$ in the Schur basis is less than or equal to the coefficient in $\mathrm{ch} H^{i+2}(\mathcal{V}(\Delta_n))$. Hence, it follows from the hard Lefschetz theorem that the sequence $(\mathrm{ch} H^{2j}(\mathcal{V}(\Delta_n)))_{0 \leq j \leq n-1}$ is palindromic and Schur-unimodal.

The Schur-unimodality of $(\mathrm{ch} H^{2j}(\mathcal{V}(\Delta_n)))_{0 \leq j \leq n-1}$ can be specialized to yield a q -analog of the unimodality of the Eulerian numbers.

Lemma 2.9 (see for example [ShWa2, Lemma 5.2]). *If f is a Schur-positive homogeneous symmetric function of degree n then*

$$(q; q)_n \mathbf{ps}(f)$$

is a polynomial in q with nonnegative coefficients.

Lemma 2.9 and the Schur-unimodality mentioned just above it, along Corollary 2.6, yield the following result.

Theorem 2.10 (Shareshian and Wachs [ShWa2, Theorem 5.3]). *For any $n \in \mathbb{P}$, the sequence $(a_{n,j}(q))_{0 \leq j \leq n-1}$ is palindromic and unimodal.*

We remark that it is not really necessary to use the hard Lefschetz theorem to establish the above mentioned unimodality results. In fact, Stembridge established a property that is stronger than Schur-unimodality by essentially manipulating the right hand side of (2.4).

Theorem 2.11 (Stembridge [Ste1]). *For any $n \in \mathbb{P}$, the sequence $(\text{ch}H^{2j}(\mathcal{V}(\Delta_n)))_{0 \leq j \leq n-1}$ is palindromic and h -unimodal.*

2.4. Smirnov words. The expression on the right hand side of (2.2) has appeared in several other contexts (see [ShWa2, Section 7]). We mention just one of these contexts here.

A Smirnov word is a word with no equal adjacent letters. Let W_n be the set of all Smirnov words of length n over alphabet \mathbb{P} . Define the enumerator

$$Y_{n,j}(x_1, x_2, \dots) := \sum_{\substack{w \in W_n \\ \text{des}(w) = j}} \mathbf{x}_w.$$

where $\mathbf{x}_w := x_{w(1)} \cdots x_{w(n)}$. (Here we are calculating $\text{des}(w)$ using the standard total order on \mathbb{P} .) A formula for the generating function of $Y_n := \sum_{j=0}^{n-1} Y_{n,j}$ was initially obtained by Carlitz, Scoville and Vaughan [CaScVa] and further studied by Dollhopf, Goulden and Greene [DoGoGr] and Stanley [St5]. Stanley obtained the following refinement of this formula.

Theorem 2.12 (Stanley (see [ShWa2, Theorem 7.2])).

$$(2.11) \quad \sum_{n,j \geq 0} Y_{n,j}(\mathbf{x}) t^j z^n = \frac{(1-t)E(z)}{E(zt) - tE(z)},$$

where $E(z) := \sum_{n \geq 0} e_n z^n$ and e_n is the elementary symmetric function of degree n .

Corollary 2.13. *For $0 \leq j \leq n-1$,*

$$\text{ch}H^{2j}(\mathcal{V}(\Delta_n)) = \omega Y_{n,j}.$$

3. RAWLINGS MAJOR INDEX AND GENERALIZED EULERIAN NUMBERS

In [Ra], Rawlings studies Mahonian permutation statistics that interpolate between the major index and the inversion index. Fix $n \in \mathbb{P}$

and $k \in [n]$. For $\sigma \in \mathfrak{S}_n$, set

$$\begin{aligned} \text{DES}_{\geq k}(\sigma) &:= \{i \in [n-1] : \sigma(i) - \sigma(i+1) \geq k\}, \\ \text{maj}_{\geq k}(\sigma) &:= \sum_{i \in \text{DES}_{\geq k}} i, \end{aligned}$$

$$\begin{aligned} \text{INV}_{< k}(\sigma) &:= \{(i, j) \in [n] \times [n] : i < j \text{ and } 0 < \sigma(i) - \sigma(j) < k\}, \\ \text{inv}_{< k}(\sigma) &:= |\text{INV}_{< k}(\sigma)|. \end{aligned}$$

Now define the *Rawlings major index* to be

$$r\text{maj}_k(\sigma) := \text{inv}_{< k}(\sigma) + \text{maj}_{\geq k}(\sigma).$$

Note that $r\text{maj}_1$ and $r\text{maj}_n$ are, respectively, the well studied major index maj and inversion number inv . Rawlings shows in [Ra] that each $r\text{maj}_k$ is a Mahonian statistic, that is,

$$(3.1) \quad \sum_{\sigma \in \mathfrak{S}_n} q^{r\text{maj}_k(\sigma)} = [n]_q!.$$

For a proof of (3.1) that is different from that of Rawlings and a generalization of (3.1) from permutations to labeled trees, see [LiWa].

In [DeSh] De Mari and Shayman introduce a class of numbers, closely related to the Rawlings major index, which they call generalized Eulerian numbers. For $k \in [n]$ and $0 \leq j$ define the De Mari-Shayman Eulerian numbers to be

$$a_{n,j}^{(k)} := |\{\sigma \in \mathfrak{S}_n : \text{inv}_{< k}(\sigma) = j\}|.$$

Note that $\text{inv}_{< 2}(\sigma) = \text{des}(\sigma^{-1})$, which implies that $a_{n,j}^{(2)} = a_{n,j}$, justifying the name “generalized Eulerian numbers”. De Mari and Shayman introduce the generalized Eulerian numbers in connection with their study of Hessenberg varieties. It follows from their work and the hard Lefschetz theorem that for each fixed $k \in [n]$, the generalized Eulerian numbers $(a_{n,j}^{(k)})_{j \geq 0}$ form a palindromic unimodal sequence of numbers.

In [ShWa4] Shareshian and Wachs consider a q -analog of the De Mari-Shayman generalized Eulerian numbers defined for $k \in [n]$ and $0 \leq j$ by

$$a_{n,j}^{(k)}(q) := \sum_{\substack{\sigma \in S_n \\ \text{inv}_{< k}(\sigma) = j}} q^{\text{maj}_{\geq k}(\sigma)}.$$

Similarly, the generalized q -Eulerian polynomials are defined by

$$A_n^{(k)}(q, t) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}_{\geq k}(\sigma)} t^{\text{inv}_{< k}(\sigma)}.$$

Now (3.1) is equivalent to,

$$A_n^{(k)}(q, q) = [n]_q!.$$

We now consider the question of whether unimodality and other properties of the generalized Eulerian numbers $a_{n,j}^{(k)}$ extend to the generalized q -Eulerian numbers $a_{n,j}^{(k)}(q)$. Although it is not at all obvious, it turns out that in the case $k = 2$, the generalized q -Eulerian numbers are equal to the q -Eulerian numbers defined in (2.8),

$$a_{n,j}^{(2)}(q) = a_{n,j}(q)$$

(see Theorem 4.17). Hence by Theorem 2.10, unimodality holds when $k = 2$. We consider this question in a more general setting, which we now describe.

For $\sigma \in \mathfrak{S}_n$ and G a graph with vertex set $[n]$, the G -inversion set of σ is

$$\text{INV}_G(\sigma) := \{(i, j) : i < j, \sigma(i) > \sigma(j) \text{ and } \{\sigma(i), \sigma(j)\} \in E(G)\}$$

and the G -inversion number is

$$\text{inv}_G(\sigma) := |\text{INV}_G(\sigma)|.$$

For $\sigma \in \mathfrak{S}_n$ and P a poset on $[n]$, the P -descent set of σ is

$$\text{DES}_P(\sigma) := \{i \in [n-1] : \sigma(i) >_P \sigma(i+1)\},$$

and the P -major index is

$$\text{maj}_P(\sigma) := \sum_{i \in \text{DES}_P(\sigma)} i.$$

Let $E(G)$ denote the edge set of a graph G . Define the *incomparability graph* $\text{inc}(P)$ of a poset P on $[n]$ to be the graph with vertex set $[n]$ and edge set $\{\{a, b\} : a \not\leq_P b \text{ and } b \not\leq_P a\}$. For $0 \leq j \leq |E(\text{inc}(P))|$ define the (q, P) -Eulerian numbers,

$$a_{P,j}(q) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{inv}_{\text{inc}(P)}(\sigma) = j}} q^{\text{maj}_P(\sigma)},$$

and the (q, P) -Eulerian polynomials

$$A_P(q, t) := \sum_{j=0}^{|E(\text{inc}(P))|} a_{P,j}(q) t^j = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}_P(\sigma)} t^{\text{inv}_{\text{inc}(P)}(\sigma)}.$$

Define $P_{n,k}$ to be the poset on vertex set $[n]$ such that $i < j$ in $P_{n,k}$ if and only if $j - i \geq k$ and let $G_{n,k}$ be the incomparability graph of $P_{n,k}$. Then

$$\text{DES}_{\geq k}(\sigma) = \text{DES}_{P_{n,k}}(\sigma)$$

and

$$\text{INV}_{< k}(\sigma) = \text{INV}_{G_{n,k}}(\sigma).$$

Hence if $P = P_{n,k}$, we have $a_{P,j}(q) = a_{n,j}^{(k)}(q)$ and $A_P(q, t) = A_n^{(k)}(q, t)$.

The unimodality property that holds for the (q, P) -Eulerian numbers in case $P := P_{n,2}$ does not hold for general P . Indeed, consider the poset P on $[3]$ whose only relation is $1 < 2$. We have

$$A_P(q, t) := (1 + q) + 2t + (1 + q^2)t^2,$$

which is neither palindromic as a polynomial in t , nor unimodal. The property $A_P(q, q) = [n]_q!$ fails as well. However there is a very nice class of posets for which unimodality seems to hold.

A *natural poset* on $[n]$ is a partial ordering of $[n]$ that is compatible with the numerical order on $[n]$; that is if a is less than b in the partial ordering then $a < b$ in the numerical ordering. A *unit interval order* is a poset that is isomorphic to a finite collection \mathcal{I} of intervals $[a, a + 1]$ on the real line, partially ordered by the relation $[a, a + 1] <_{\mathcal{I}} [b, b + 1]$ if $a + 1 < b$. Natural unit interval orders on $[n]$ have the following characterization.

Proposition 3.1. *The poset P is a natural unit interval order if and only if for all $x < y < z \in [n]$, neither the disjoint union $\{x <_P y\} \oplus \{z\}$ nor the disjoint union $\{x\} \oplus \{y <_P z\}$ is an induced subposet of P .*

The poset $P_{n,k}$ is an example of a natural unit interval order.

It is a consequence of a result of Kasraoui [Ka, Theorem 1.8] that if P is a natural unit interval order then $\text{inv}_{\text{inc}(P)} + \text{maj}_P$ is Mahonian. In other words, if P is a natural unit interval order on $[n]$ then

$$A_P(q, q) = [n]_q!.$$

From the work of De Mari, Procesi and Shayman [DePrSh] we have the following result. The proof is discussed in Section 5.

Theorem 3.2. *Let P be a natural unit interval order. Then the P -Eulerian polynomial $A_P(1, t)$ is palindromic and unimodal.*

It is easy to verify palindromicity of the q -analog $A_P(q, t)$. Unimodality for the q -analog (generalizing Theorem 2.10) seems to be more difficult.

Conjecture 3.3 (Shareshian and Wachs [ShWa4]). *Let P be a natural unit interval order on $[n]$. Then the sequence $(a_{P,j}(q))_{0 \leq j \leq |E(\text{inc}(P))|}$ is unimodal.*

The following conjecture generalizes Theorem 2.7.

Conjecture 3.4 (Shareshian and Wachs [ShWa4]). *Let P be a natural unit interval order on $[n]$. If $dm = n$ then there is a polynomial $B_{P,d}(t) \in \mathbb{N}[t]$ such that*

$$A_P(\xi_d, t) = B_{P,d}(t)[d]_t^m.$$

Moreover, the palindromic sequence $(a_{P,j}(\xi_d))_{0 \leq j \leq |E(\text{inc}(P))|}$ is a unimodal sequence of positive integers.

In addition to the case $P = P_{n,2}$ (Theorems 2.10 and 2.7), these conjectures have been verified for $P_{n,k}$ when $k = 1, n-2, n-1, n$, and by computer for all k when $n \leq 8$. An approach to proving them in general will be presented in Sections 4 and 5.

4. CHROMATIC QUASISYMMETRIC FUNCTIONS

4.1. Stanley's chromatic symmetric functions. Let G be a graph with vertex set $[n]$ and edge set $E = E(G) \subseteq \binom{[n]}{2}$. A *proper \mathbb{P} -coloring* of G is a function c from $[n]$ to the set \mathbb{P} of positive integers such that whenever $\{i, j\} \in E$ we have $c(i) \neq c(j)$. Given any function $c : [n] \rightarrow \mathbb{P}$, set

$$x_c := \prod_{i=1}^n x_{c(i)}.$$

Let $C(G)$ be the set of proper \mathbb{P} -colorings of G . In [St5], Stanley defined the *chromatic symmetric function* of G ,

$$X_G(\mathbf{x}) := \sum_{c \in C(G)} x_c.$$

It is straightforward to confirm that $X_G \in \Lambda_{\mathbb{Z}}$. The chromatic symmetric function is a generalization of the chromatic polynomial $\chi_G : \mathbb{P} \rightarrow \mathbb{P}$, where $\chi_G(n)$ is the number of proper colorings of G with n colors. Indeed, $X_G(1^n) = \chi_G(n)$, where $X_G(1^n)$ is the specialization of $X_G(\mathbf{x})$ obtained by setting $x_i = 1$ for $1 \leq i \leq n$ and $x_i = 0$ for $i > n$. Chromatic symmetric functions are studied in various papers, including [St5, St6, Ga, Ch, MaMoWa].

We recall Stanley's description of the power sum decomposition of $X_G(\mathbf{x})$, for arbitrary G with vertex set $[n]$. We call a partition $\pi = \pi_1 | \dots | \pi_l$ of $[n]$ into nonempty subsets G -*connected* if the subgraph of G induced on each block π_i of π is connected. The set $\Pi_{G,n}$ of all

G -connected partitions of $[n]$ is partially ordered by refinement (that is, $\pi \leq \theta$ if each block of π is contained in some block of θ). We write μ_G and $\hat{0}$, respectively, for the Möbius function on $\Pi_{G,n}$, and the minimum element $1|\dots|n$ of $\Pi_{G,n}$. For $\pi \in \Pi_{G,n}$, we write $\text{par}(\pi)$ for the partition of n whose parts are the sizes of the blocks of π .

Theorem 4.1 (Stanley [St5, Theorem 2.6]). *Let G be a graph with vertex set $[n]$. Then*

$$(4.1) \quad \omega X_G(\mathbf{x}) = \sum_{\pi \in \Pi_{G,n}} |\mu_G(\hat{0}, \pi)| p_{\text{par}(\pi)}.$$

Consequently $\omega X_G(\mathbf{x})$ is p -positive.

A poset P is called $3+1$ -free if there are no four elements $w, x, y, z \in P$ such that the only relations between these elements are $x < y < z$. Similarly, P is $2+2$ -free if there are no four elements $a, b, c, d \in P$ such that the only relations between the elements are $a < b$ and $c < d$. A classical result (see [ScSu]) says that a poset is a unit interval order if and only if it is both $3+1$ -free and $2+2$ -free.

Conjecture 4.2 (Stanley [St5, Conjecture 5.1]). *Let G be the incomparability graph of a $3+1$ -free poset. Then $X_G(\mathbf{x})$ is e -positive.*

This conjecture is still open even for unit interval orders. Gasharov [Ga] obtains the weaker, but still very interesting, result that, under the assumptions of Conjecture 4.2, $X_G(\mathbf{x})$ is Schur-positive. Let P be a poset on n and let λ be a partition of n . Gasharov defines a P -tableau of shape λ to be a filling of a Young diagram of shape λ (in English notation) with elements of P such that

- each element of P appears exactly once,
- if $y \in P$ appears immediately to the right of $x \in P$, then $y >_P x$, and
- if $y \in P$ appears immediately below $x \in P$, then $y \not\prec_P x$.

Given a P -tableau T , let $\lambda(T)$ be the shape of T . Let \mathcal{T}_P be the set of all P -tableaux.

Theorem 4.3 (Gasharov [Ga]). *Let P be a $3+1$ -free poset. Then*

$$X_{\text{inc}(P)}(x) = \sum_{T \in \mathcal{T}_P} s_{\lambda(T)}.$$

4.2. A quasisymmetric refinement. Shreshian and Wachs [ShWa4] define the *chromatic quasisymmetric function* of G as

$$X_G(\mathbf{x}, t) := \sum_{c \in C(G)} t^{\text{asc}(c)} x_c,$$

where

$$\text{asc}(c) := |\{\{i, j\} \in E(G) : i < j \text{ and } c(i) < c(j)\}|.$$

It is straightforward to confirm that $X_G(x, t) \in \mathcal{Q}_{\mathbb{Z}}[t]$.

While the chromatic quasisymmetric function is defined for an arbitrary graph on vertex set $[n]$, our results will concern incomparability graphs of partially ordered sets. The natural unit interval orders are particularly significant in the theory of chromatic quasisymmetric functions because they yield symmetric functions.

Proposition 4.4 (Shareshian and Wachs [ShWa4]). *Let P be a natural unit interval order. Then*

$$X_{\text{inc}(P)}(\mathbf{x}, t) \in \Lambda_{\mathbb{Z}}[t].$$

Example 4.5. $P = P_{n,1}$. The incomparability graph has no edges. Hence

$$(4.2) \quad X_{\text{inc}(P_{n,1})}(\mathbf{x}, t) = e_1^n$$

Example 4.6. $P = P_{n,n}$. The incomparability graph is the complete graph. Each proper coloring c is an injective map which can be associated with a permutation $\sigma \in \mathfrak{S}_n$ for which $\text{inv}(\sigma) = \text{asc}(c)$. It follows that

$$(4.3) \quad X_{\text{inc}(P_{n,n})}(\mathbf{x}, t) = e_n \sum_{\sigma \in \mathfrak{S}_n} t^{\text{inv}(\sigma)} = [n]_t! e_n$$

Example 4.7. $P = P_{n,2}$. Recall that the incomparability graph of $P_{n,2}$ is a path. To each proper coloring c of the path $\text{inc}(P_{n,2})$ one can associate the word $w(c) := c(n), c(n-1), \dots, c(1)$. This word is clearly a Smirnov word of length n (c.f. Section 2.4) and $\text{des}(w(c)) = \text{asc}(c)$. Since w is a bijection from $C(\text{inc}(P_{n,2}))$ to W_n , we have

$$X_{\text{inc}(P_{n,2})}(\mathbf{x}, t) = \sum_{j=0}^{n-1} Y_{n,j}(\mathbf{x}) t^j.$$

It therefore follows from Corollary 2.13 that

$$(4.4) \quad \omega X_{\text{inc}(P_{n,2})}(\mathbf{x}, t) = \sum_{j=0}^{n-1} \text{ch} H^{2j}(\mathcal{V}(\Delta_n)) t^j.$$

From (4.2), (4.3), (4.4) and Theorem 2.11 we see that the following conjectured refinement of the unit interval order case of Stanley's conjecture (Conjecture 4.2) is true for $P = P_{n,k}$ when $k = 1, 2, n$.

Conjecture 4.8 (Shareshian and Wachs [ShWa4]). *Let P be a natural unit interval order. Then $X_{\text{inc}(P)}(\mathbf{x}, t)$ is a palindromic, e -positive, e -unimodal polynomial in t .*

Palindromicity is easy to prove in general. We have verified e -positivity and e -unimodality for several other cases including $P = P_{n,k}$ when $k = n-1, n-2$, and by computer for all $n \leq 8$. Our computation yields,

(4.5)

$$X_{\text{inc}(P_{n,n-1})}(\mathbf{x}, t) = [n-2]_t! ([n]_t[n-2]_t e_n + t^{n-2} e_{n-1,1})$$

(4.6)

$$X_{\text{inc}(P_{n,n-2})}(\mathbf{x}, t) = [n-4]_t! ([n]_t[n-3]_t^3 e_n + [n-2]_t t^{n-3} ([n-3]_t + [2]_t[n-4]_t) e_{n-1,1} + t^{2n-7} [2]_t e_{n-2,2}),$$

from which the conjecture is easily verified using Proposition 1.3.

One more bit of evidence for e -positivity of $X_{\text{inc}(P)}(\mathbf{x}, t)$ is given by Theorem 4.9 below. We say that $\sigma \in \mathfrak{S}_n$ has a left-to-right P -maximum at $r \in [n]$ if $\sigma(r) >_P \sigma(s)$ for $1 \leq s < r$. The left-to-right P -maximum at 1 will be referred to as trivial. (The notion of (trivial) left-to-right P -maximum can be extended in an obvious way to permutations of any subset of P .) Let

$$(4.7) \quad c_P(t) := \sum_{\sigma} t^{\text{inv}_{\text{inc}(P)}(\sigma)},$$

where σ ranges over the permutations in \mathfrak{S}_n with no descents and no nontrivial left-to-right P -maxima.

Theorem 4.9 (Shareshian and Wachs [ShWa4]). *Let P be a natural unit interval order on $[n]$. The coefficient of e_n in the expansion of $X_{\text{inc}(P)}(\mathbf{x}, t)$ in the e -basis of $\Lambda_{\mathbb{Z}[t]}$ is equal to $c_P(t)$.*

Conjecture 4.10 ([Shareshian and Wachs [ShWa4]]). *Let P be a natural unit interval order on $[n]$. Then*

$$c_P(t) = [n]_t \prod_{i=2}^n [a_i]_t,$$

where $a_i = |\{j \in [i-1] : \{i, j\} \in E(\text{inc}(P))\}|$. Consequently $c_P(t)$ is palindromic and unimodal.

In the case that $P = P_{n,2}$, this conjecture is true by (2.4) and (4.4). It is also true for $P_{n,k}$ when $k = 1, n, n-1, n-2$ by (4.2), (4.3), (4.5) and (4.6), respectively.

4.3. Schur and power sum decompositions. When P is a natural unit interval order, we have the following refinement of Gasharov's Schur positivity result (Theorem 4.3). For a P -tableau T and a graph G with vertex set $[n]$, let $\text{inv}_G(T)$ be the number of edges $\{i, j\} \in E(G)$ such that $i < j$ and i appears to the south of j in T .

Theorem 4.11 (Shareshian and Wachs [ShWa4]). *Let P be a natural unit interval order poset on $[n]$ and let G be the incomparability graph of P . Then*

$$X_G(x, t) = \sum_{T \in \mathcal{T}_P} t^{\text{inv}_G(T)} s_{\lambda(T)}.$$

By 4.4 we have the following.

Corollary 4.12. *Let $\lambda \vdash n$. The multiplicity of the irreducible Specht module S^λ in $H^{2j}(\mathcal{V}(\Delta_n))$ is equal to the number of $P_{n,2}$ -tableaux of shape λ' with $\text{inv}_{\text{inc}(P_{n,2})}(T) = j$.*

By comparing this decomposition to Stembridge's decomposition (Theorem 2.2) we see that the number of marked tableaux of shape λ and index j equals the number of $P_{n,2}$ -tableaux of shape λ' and $\text{inv}_{\text{inc}(P_{n,2})}(T) = j$. It would be interesting to find a bijective proof of this fact.

Theorem 4.11 shows that when G is the incomparability graph of a natural unit interval order, the coefficient of each power of t in $\omega X_G(\mathbf{x}, t)$ is a nonnegative integer combination of Schur functions and therefore the Frobenius characteristic of an actual representation of \mathfrak{S}_n . Conjecture 4.8 says that this linear representation arises from a permutation representation in which each point stabilizer is a Young subgroup. In Section 5 we present a very promising concrete candidate for the desired permutation representation.

Next we attempt to refine Stanley's p -basis decomposition of the chromatic symmetric functions (Theorem 4.1). In Conjectures 4.13 and 4.14 below, we provide two proposed formulae for this power sum decomposition. With $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_l)$ a partition of n and P a natural unit interval order on $[n]$, we call a permutation $\sigma \in \mathfrak{S}_n$, $(P, \mu, 1)$ -compatible if, when we break σ (in one line notation) into consecutive segments of lengths μ_1, \dots, μ_l , the segments have no P -descents and no nontrivial left-to-right P -maxima. We call $\sigma \in \mathfrak{S}_n$, $(P, \mu, 2)$ -compatible if, when we break σ into consecutive segments of lengths μ_1, \dots, μ_l , the segments have no P -ascents and they begin with the numerically smallest letter of the segment. We write $\mathfrak{S}_{P, \mu, i}$, where $i = 1, 2$, for the set of all (P, μ, i) -compatible elements of \mathfrak{S}_n .

Conjecture 4.13 (Shareshian and Wachs [ShWa4]). *Let P be a natural unit interval order on $[n]$ with incomparability graph G . Then*

$$\omega X_G(\mathbf{x}, t) = \sum_{\mu \vdash n} z_\mu^{-1} p_\mu \sum_{\sigma \in \mathfrak{S}_{P, \mu, 1}} t^{\text{inv}_G(\sigma)}.$$

Conjecture 4.14 (Shareshian and Wachs [ShWa4]). *Let P be a natural unit interval order on $[n]$ with incomparability graph G . Then*

$$\omega X_G(\mathbf{x}, t) = \sum_{\mu \vdash n} z_\mu^{-1} p_\mu \prod_{i=1}^{l(\mu)} [\mu_i] \sum_{\sigma \in \mathfrak{S}_{P, \mu, 2}} t^{\text{inv}_G(\sigma)}.$$

It can be shown that both conjectures reduce to the formula in Theorem 4.1 when $t = 1$. Conjecture 4.13 says that the coefficient of $z_{(n)}^{-1} p_{(n)}$ is equal to $c_P(t)$ defined in (4.7). Since the coefficient of $h_{(n)}$ in the h -basis decomposition must equal the coefficient of $z_{(n)}^{-1} p_{(n)}$ in p -basis decomposition, by Theorem 4.9, Conjecture 4.13 gives the correct coefficient of $z_{(n)}^{-1} p_{(n)}$. When $P = P_{n,2}$ it is not difficult to see that both conjectures reduce to Theorem 2.3.

4.4. Fundamental quasisymmetric function basis decomposition.

Theorem 4.15 (Shareshian and Wachs [ShWa4]). *Let G be the incomparability graph of a poset P on $[n]$. Then*

$$\omega X_G(\mathbf{x}, t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{inv}_G(\sigma)} F_{n, \text{DES}_P(\sigma)},$$

where ω is the involution on $\mathcal{Q}_{\mathbb{Z}}$ that maps $F_{n,S}$ to $F_{n, [n-1] \setminus S}$ for each $n \in \mathbb{N}$ and $S \subseteq [n-1]$. (This extends the involution ω on $\Lambda_{\mathbb{Z}}$ that maps h_n to e_n)

Theorem 4.15 refines Corollary 2 in Chow's paper [Ch]. Indeed, one obtains Chow's result by setting $t = 1$ in Theorem 4.15. Our proof of Theorem 4.15 follows the same path as Chow's proof of his Corollary 2.

We use Theorem 4.15 and (2.7) to compute the principal stable specialization of $\omega X_G(\mathbf{x}, t)$.

Corollary 4.16. *Let G be the incomparability graph of a poset P on $[n]$. Then*

$$(q; q)_n \text{ps}(\omega X_G(\mathbf{x}, t)) = A_P(q, t)$$

Recall that for $P = P_{n,2}$ we have a formulation for the expansion of $\omega X_{\text{inc}(P)}(\mathbf{x}, t)$ in the fundamental quasisymmetric function basis that is

different from that of Theorem 4.15. It is obtained by combining (2.6) and (2.11). By equating these formulations, we obtain the identity,

$$(4.8) \quad \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma^{-1})} F_{n, \text{DES}_{\geq 2}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)} F_{n, \text{DEX}(\sigma)}.$$

Taking the stable principle specialization of both sides of (4.8) yields the following new Euler-Mahonian result.

Theorem 4.17 (Shareshian and Wachs [ShWa4]). *For all $n \in \mathbb{P}$,*

$$\sum_{\sigma \in \mathfrak{S}_n} q^{r\text{maj}_2(\sigma)} t^{\text{des}(\sigma^{-1})} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} t^{\text{exc}(\sigma)}$$

It would be interesting to find bijective proofs of (4.8) and Theorem 4.17.

It follows from Corollary 4.16 and Lemma 2.9 that Conjecture 3.3 asserting the unimodality of $A_P(q, t)$ is a consequence of Conjecture 4.8. By taking a specialization called (nonstable) principal specialization, a stronger unimodality assertion follows from Conjecture 3.3. For a poset P on $[n]$ and $\sigma \in \mathfrak{S}_n$, define $\text{des}_P(\sigma) := |\text{DES}_P(\sigma)|$.

Conjecture 4.18. *Let P a natural unit interval order on $[n]$. Then*

$$A_P(q, p, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}_P(\sigma)} p^{\text{des}_P(\sigma)} t^{\text{inv}_{\text{inc}(P)}(\sigma)}$$

is a palindromic unimodal polynomial in t .

It follows from Corollary 4.16 and Lemma 2.8 that $A_P(\xi_d, t)$ is the coefficient of $z_{dm}^{-1} p_{dm}$ in the p -basis expansion of $\omega X_{\text{inc}(P)}(\mathbf{x}, t)$ when P is a natural unit interval order on $[dm]$. Hence Conjecture 3.4 on the evaluation of $A_P(q, t)$ at roots of unity is a consequence of Conjectures 4.8 and 4.14.

5. HESSENBERG VARIETIES

Let $G = GL_n(\mathbb{C})$ and let B be the set of upper triangular matrices in G . The (type A) *flag variety* is the quotient space G/B . Fix now a natural unit interval order P and let $M_{n,P}$ be the set of all $n \times n$ complex matrices (a_{ij}) such that $a_{ij} = 0$ whenever $i >_P j$. Fix a nonsingular diagonal $n \times n$ matrix s with n distinct eigenvalues. The *regular semisimple Hessenberg variety of type A* associated to P is

$$\mathcal{H}(P) := \{gB \in G/B : g^{-1}sg \in M_{n,P}\}.$$

(Note that $\mathcal{H}(P)$ is well defined, since the group B normalizes the set $M_{n,P}$.)

Certain regular semisimple Hessenberg varieties of type A were studied initially by De Mari and Shayman in [DeSh]. Hessenberg varieties for other Lie types are defined and studied by De Mari, Procesi and Shayman in [DePrSh]. Such varieties are determined by certain subsets of a root system. In [DePrSh, Theorem 11] it is noted that, for arbitrary Lie type, the Hessenberg variety associated with the set of simple roots is precisely the toric variety associated with the corresponding Coxeter complex. In particular, in Lie type A, the poset giving rise to the regular semisimple Hessenberg variety associated with simple roots is $P_{n,2}$ since $\text{inc}(P_{n,2}) = \{\{i, i+1\} : i \in [n-1]\}$. Thus

$$(5.1) \quad \mathcal{H}(P_{n,2}) = \mathcal{V}(\Delta_n).$$

De Mari, Procesi and Shayman also show that the cohomology of $\mathcal{H}(P)$ is concentrated in even degrees and that for $0 \leq j \leq |E(\text{inc}(P))|$,

$$(5.2) \quad \dim H^{2j}(\mathcal{H}(P)) = a_{P,j}(1).$$

Hence unimodality of the P -Eulerian polynomials $A_P(1, t)$ (Theorem 3.2) follows from the hard Lefschetz theorem since $\mathcal{H}(P)$ is smooth. Our approach to establishing the unimodality of the (q, P) -Eulerian polynomial $A_P(q, t)$ is to find a representation of the symmetric group on $H^{2j}(\mathcal{H}(P))$ such that the stable principal specialization of its Frobenius characteristic yields $a_{P,j}(q)$. If the hard Lefschetz map commutes with the action of the symmetric group then it follows from Schur's Lemma that $A_P(q, t)$ is unimodal.

For the cases $P = P_{n,k}$, $k = 1, 2, n$, there are natural well understood representations of \mathfrak{S}_n on the cohomology of $\mathcal{H}(P)$ on which our approach works. Indeed, $\mathcal{H}(P_{n,1})$ consists of $n!$ isolated points and the representation of \mathfrak{S}_n on $H^0(\mathcal{H}(P_{n,1}))$ is the regular representation. When $k = 2$, the representation of \mathfrak{S}_n on $H^*(\mathcal{H}(P_{n,k})) = H^*(\mathcal{V}(\Delta_n))$ is the representation discussed in Section 2. When $k = n$, one gets the representation of \mathfrak{S}_n on the cohomology of the complete flag variety studied by Tymoczko in [Ty2]. For $0 \leq j \leq n$, this representation on $H^{2j}(\mathcal{H}(P_{n,n}))$ is given by $a_{n,j}^{(n)}$ copies of the trivial representation.

For general natural unit interval orders P , Tymoczko [Ty2] has defined a representation of \mathfrak{S}_n on $H^*(\mathcal{H}(P))$ via a theory of Goresky, Kottwitz and MacPherson known as GKM theory (see [GoKoMacP], and see [Ty1] for an introductory description of GKM theory.) The centralizer $T = C_G(s)$ consists of the diagonal matrices in G . It follows that the torus T acts (by left translation) on $\mathcal{H}(P)$. The technical conditions required for application of GKM theory are satisfied by this action, and it follows that one can describe the cohomology of $\mathcal{H}(P)$ using the moment graph associated to this action. The moment graph

M is a subgraph of the Bruhat graph. The vertex set of M is \mathfrak{S}_n and the edges connect pairs of elements that differ by a transposition (i, j) such that $i >_P j$. Thus M admits an action of \mathfrak{S}_n , and this action can be used to define a linear representation of \mathfrak{S}_n on the cohomology $H^*(\mathcal{H}(P))$.

Tymoczko's representation of \mathfrak{S}_n on $H^*(\mathcal{H}(P))$, in each of the cases $P = P_{n,k}$, $k = 1, 2, n$, is the same as the respective representations of \mathfrak{S}_n discussed above.

By applying the hard Lefschetz theorem, MacPherson and Tymoczko obtain the following result.

Theorem 5.1 (MacPherson and Tymoczko [MacTy]). *For all natural unit interval orders P , the sequence $(\text{ch} H^{2j}(\mathcal{H}(P)))_{0 \leq j \leq |E(\text{inc}(P))|}$ is palindromic and Schur-unimodal.*

Tymoczko poses the following problem.

Problem 5.2 (Tymoczko [Ty2]). *Given any natural unit interval order P on vertex set $[n]$, describe the decomposition of the representation of \mathfrak{S}_n on $H^{2j}(\mathcal{H}(P))$ into irreducibles.*

We finally come to the conjecture that ties together the three topics of this paper.

Conjecture 5.3 (Shareshian and Wachs [ShWa4]). *Let P be a natural unit interval order on $[n]$ with incomparability graph G . Then*

$$(5.3) \quad \sum_{j=0}^{|E(G)|} \text{ch} H^{2j}(\mathcal{H}(P)) t^j = \omega X_{\text{inc}(P)}(\mathbf{x}, t)$$

By (5.1) and (4.4) the conjecture is true in the case that $P = P_{n,2}$. It is straightforward to verify in the case $P = P_{n,k}$, when $k = 1, n-1, n$, and for all natural unit interval orders on $[n]$ when $n \leq 4$. It follows from (5.2) that the coefficient of the monomial symmetric function m_{1^n} in the expansion of the left hand side of (5.3) in the monomial symmetric function basis equals the coefficient of m_{1^n} in the expansion of the right hand side of (5.3).

Conjecture 5.3, if true, would have many important ramifications. It would allow us to transfer what we know about chromatic quasisymmetric functions to Tymoczko's representation and vice-versa. For instance Theorem 4.11 would provide a solution to Tymoczko's problem. Conjecture 5.3 provides an approach to proving Stanley's conjecture (Conjecture 4.2) for unit interval orders. Indeed one would only have to prove the following conjecture.

Conjecture 5.4 (Sharesian and Wachs [ShWa4]). *For all natural interval orders P , Tymoczko's representation of \mathfrak{S}_n on $H^*(\mathcal{H}(P))$ is a permutation representation in which each point stabilizer is a Young subgroup.*

Our refinement of Stanley's conjecture (Theorem 4.8) would be equivalent to the following strengthening of the result of MacPherson and Tymoczko.

Conjecture 5.5 (Sharesian and Wachs [ShWa4]). *For all natural unit interval orders P , the palindromic sequence $(\text{ch}H^{2j}(\mathcal{H}(P)))_{0 \leq j \leq |E(\text{inc}(P))|}$ is h -unimodal.*

To establish the unimodality of the (q, P) -Eulerian numbers (Conjecture 3.3) one would only need to prove Conjecture 5.3. Indeed Conjecture 5.3, Theorem 5.1, Corollary 4.16 and Lemma 2.9 imply the unimodality result. The more general unimodality conjecture for $A_P(q, p, t)$ (Conjecture 4.18) is also a consequence of Conjecture 5.3.

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REFERENCES

- [CaScVa] L. Carlitz, R. Scoville, and T. Vaughan, Enumeration of pairs of sequences by rises, falls and levels, *Manuscripta Math.* **19** (1976), 211-243.
- [Ch] T. Chow, Descents, quasi-symmetric functions, Robinson-Schensted for posets, and the chromatic symmetric function, *J. Algebraic Combin.* **10** (1999), no. 3, 227-240.
- [DePrSh] F. De Mari, C. Procesi and M. Shayman, Hessenberg varieties, *Trans. Amer. Math. Soc.* **332** (1992), no. 2, 529-534.
- [DeSh] F. De Mari and M. Shayman, Generalized Eulerian numbers and the topology of the Hessenberg variety of a matrix, *Acta Appl. Math.* **12** (1988), no. 3, 213-235.
- [De] J. Désarménien, Fonctions symétriques associées à des suites classiques de nombres, *Ann. scient. Éc. Norm. Sup.* **16** (1983) 271-304.
- [DoLu] I. Dolgachev and V. Lunts, A character formula for the representation of a Weyl group in the cohomology of the associated toric variety, *J. Algebra* **168** (1994), 741-772.
- [DoGoGr] J. Dollhopf, I. Goulden, and C. Greene, Words avoiding a reflexive acyclic relation, *Electron. J. Combin.* **11** (2006) #R28.
- [FoHa1] D. Foata and G.-N. Han, Fix Mahonian calculus III; A quadruple distribution, *Monatshefte für Mathematik* **154** (2008), 177-197.

- [FoHa2] The q -tangent and q -secant numbers via basic Eulerian polynomials, *Proc. Amer. Math. Soc.* **138** (2010) 385-393.
- [Ga] V. Gasharov, Incomparability graphs of $(3 + 1)$ -free posets are s -positive, Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994), *Discrete Math.* **157** (1996), no. 1-3, 193-197.
- [GoKoMacP] M. Goresky, R. Kottwitz and R. MacPherson, Equivariant cohomology, Koszul duality, and the localization theorem, *Invent. Math.* **131** (1998), no. 1, 25-83.
- [HeWa] A. Henderson and M.L. Wachs, Unimodality of Eulerian quasisymmetric functions, preprint, arXiv:1101.444.
- [Ka] A. Kasraoui, A classification of Mahonian maj-inv statistics. *Adv. in Appl. Math.* **42** (2009), no. 3, 342-357.
- [Le] G. I. Lehrer, Rational points and Coxeter group actions on the cohomology of toric varieties, *Ann. Inst. Fourier (Grenoble)* **58** (2008), no. 2, 671-688.
- [LiWa] K. Liang and M.L. Wachs, Mahonian statistics on labeled forests, *Discrete Math.* **99** (1992), 181-197.
- [MacTy] R. MacPherson and J. Tymoczko, in preparation.
- [MaMoWa] J. Martin, M. Morin and J. Wagner, On distinguishing trees by their chromatic symmetric functions, *J. Combin. Theory Ser. A* **115** (2008), no. 2, 237-253.
- [Pr] C. Procesi, The toric variety associated to Weyl chambers, in *Mots*, 153-161, Lang. Raison. Calc., Herms, Paris, 1990.
- [Ra] D. Rawlings, The r -major index, *J. Combin. Theory Ser. A* **31** (1981), no. 2, 175-183.
- [SaShWa] B. Sagan, J. Shareshian and M. L. Wachs, Eulerian quasisymmetric functions and cyclic sieving, *Adv. Applied Math* **46** (2011), 536-562.
- [ScSu] D. Scott and P. Suppes, Foundational aspects of theories of measurement, *J. Symb. Logic* **23** (1958), 113-128.
- [ShWa1] J. Shareshian and M. L. Wachs, q -Eulerian polynomials: excedance number and major index, *Electron. Res. Announc. Amer. Math. Soc.* **13** (2007), 33-45.
- [ShWa2] J. Shareshian and M. L. Wachs, Eulerian quasisymmetric functions, *Adv. Math.* **225** (2010), no. 6, 2921-2966.
- [ShWa3] Poset homology of Rees products and q -Eulerian polynomials, *Elect. J. Combin.* **16** (2009), R20.
- [ShWa4] J. Shareshian and M. L. Wachs, Chromatic quasisymmetric functions, *in preparation*.
- [St1] R. P. Stanley, Combinatorial Applications of the Hard Lefschetz Theorem, *Proceedings of the International Congress of Mathematicians* 1983, Warszawa.
- [St2] R. P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, *Graph theory and its applications: East and West (Jinan, 1986)*, 500-535, Ann. New York Acad. Sci., **576**, New York Acad. Sci., New York, 1989.

- [St3] R. P. Stanley, *Enumerative Combinatorics, Volume 1*, with a foreword by Gian-Carlo Rota, Cambridge Studies in Advanced Mathematics, **49**, Cambridge University Press, Cambridge, 1997.
- [St4] R. P. Stanley, *Enumerative Combinatorics, Volume 2*, with a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin, Cambridge Studies in Advanced Mathematics, **62**, Cambridge University Press, Cambridge, 1999.
- [St5] R. P. Stanley, A symmetric function generalization of the chromatic polynomial of a graph, *Adv. Math.* **111** (1995), no. 1, 166-194.
- [St6] R. P. Stanley, Graph colorings and related symmetric functions: ideas and applications: a description of results, interesting applications, & notable open problems, Selected papers in honor of Adriano Garsia (Taormina, 1994), *Discrete Math.* **193** (1998), no. 1-3, 267-286.
- [Ste1] J.R. Stembridge, Eulerian numbers, tableaux, and the Betti numbers of a toric variety, *Discrete Math.* **99** (1992), 307-320.
- [Ste2] J.R. Stembridge, Some permutation representations of Weyl groups associated with the cohomology of toric varieties, *Adv. Math.* **106** (1994), 244-301.
- [Ty1] J. Tymoczko, An introduction to equivariant cohomology and homology, following Goresky, Kottwitz, and MacPherson, Snowbird lectures in algebraic geometry, 169-188, *Contemp. Math.*, **388**, Amer. Math. Soc., Providence, RI, 2005.
- [Ty2] J. Tymoczko, Permutations actions on equivariant cohomology, Toric topology, 365-384, *Contemp. Math.* **460**, Amer. Math. Soc., Providence, RI, 2008.

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